Notions of properness for semigroups

York Semigroup
3rd December 2014

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A Farewell to CAUL

....originally presented as part of a Farewell to Centro de Algebra da Universidade de Lisboa

Gracinda

Notions of properness for semigroups

1. Notions of properness for **which** semigroups? Ehresmann semigroups.
2. The classical background.
3. Some candidates for propriety.
4. Using one candidate: S-labelled trees.

The set of idempotents of any semigroup S is denoted by $E(S)$.

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1. Ehresmann semigroups: Unary and biunary semigroups

A **unary** semigroup is a semigroup equipped with a unary operation, normally denoted by

$$a \mapsto a^+.$$

A **biunary** semigroup is a semigroup equipped with two unary operations, normally denoted by

$$a \mapsto a^+ \text{ and } a \mapsto a^*.$$

We regard unary [biunary] semigroups as algebras with signature $(2, 1)$ $[(2, 1, 1)]$.

Similarly for unary and biunary **monoids**.

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1. Ehresmann semigroups: Inverse semigroups

A semigroup S is **inverse** if for each $a \in S$ there exists a unique $a' \in S$ such that

$$a = aa'a \text{ and } a' = a'aa'.$$

If S is inverse, then for any $a \in S$ we have $aa', a'a \in E(S)$ and

$$ef = fe \text{ for all } e, f \in E(S).$$

It follows that $E(S)$ is a **semilattice** i.e. a commutative semigroup of idempotents.

A semilattice is **partially ordered** under

$$e \leq f \text{ if and only if } ef = e$$

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Clearly, an inverse semigroup is a unary semigroup under

$$a \mapsto a'.$$

An inverse semigroup is also **biunary** where

$$a \mapsto a^{+} = aa' \text{ and } a \mapsto a^{*} = a'a.$$

1. Left Ehresmann semigroups: A variety of unary semigroups

Definition A unary semigroup $(S, \cdot, ^+)$ is **left Ehresmann** if it satisfies the identities Σ_ℓ:

$$a^+a^+ = a^+,\ a^+b^+ = b^+a^+,\ (a^+b^+)^+ = a^+b^+,\ a^+a = a,\ (ab^+)^+ = (ab)^+.$$

Let

$$E = \{a^+ : a \in S\}.$$

Then E is a **semilattice**, the semilattice of **projections** of S.

Example 1 Inverse semigroups are left Ehresmann under $a^+ = aa'$.

Example 2 Any monoid is left Ehresmann with $a^+ = 1$ for all $a \in M$. It is a **reduced** left Ehresmann semigroup.

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$$(a^*)^+ = a^*, \ (a^+)^* = a^+.$$

If S is Ehresmann then

$$E = \{a^* : a \in S\} = \{a^+ : a \in S\}.$$

Example 1 Inverse semigroups are Ehresmann under $a^+ = aa'$ and $a^* = a'a$.

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1. Ehresmann semigroups: Observations, examples

- The name 'Ehresmann' was coined by Lawson, 1991; he first established the connection between Ehresmann semigroups and the bi-ordered categories of C. Ehresmann.
- Inverse semigroups are Ehresmann and **inverse semigroups are important!!**
- As Ehresmann semigroups are varieties, they are closed under H,S,P; free algebras exist.
- Any biunary subsemigroup of an inverse semigroup is Ehresmann.
- Type A semigroups (later called ample) are Ehresmann; restriction semigroups are Ehresmann.
- Ehresmann semigroups are the variety generated by the quasi-variety of adequate semigroups.
- Above in one-sided case and monoid case.

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1. Ehresmann semigroups: Observations, examples

- $\mathcal{PT}_X$ is Ehresmann where α^+ (α^*) is the identity map in the domain (range) of α; in fact, $\mathcal{PT}_X$ is **left restriction** Trokhimenko, 1973.
- $\mathcal{B}_X$ is Ehresmann under

$$\rho^+ = \{(a,a) : a \in \text{dom } \rho\} \text{ and } \rho^* = \{(a,a) : a \in \text{im } \rho\}.$$

- Any semidirect product $Y \rtimes M$, where Y is a semilattice and M a monoid is left restriction, hence left Ehresmann.
- Let Y be a semilattice. Then the **free idempotent generated semigroup** $\mathrm{IG}(Y)$ is adequate, hence Ehresmann. G, Yang, 2013.

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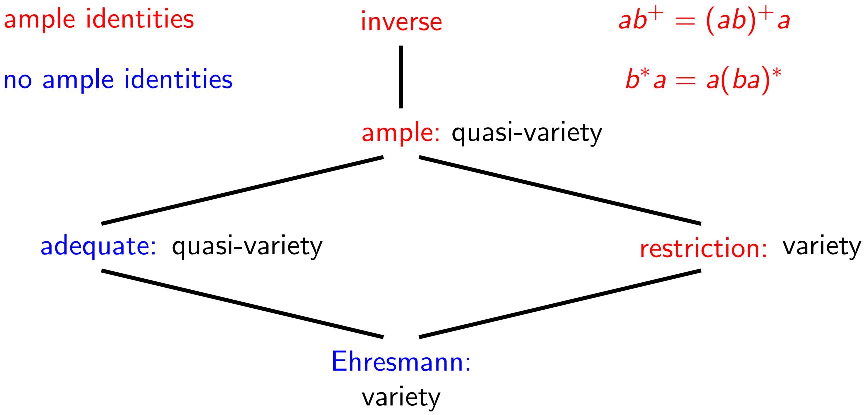
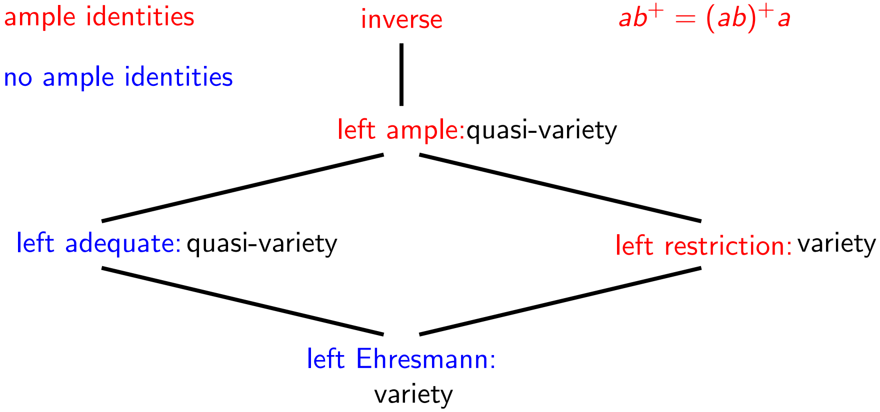
1. Ehresmann semigroups: The bigger picture: Classes of biunary semigroups with semilattices of idempotents



1. Left Ehresmann semigroups: The bigger picture: Classes of unary semigroups with semilattices of idempotents



Let S be an inverse semigroup.

- $\sigma = \langle E(S) \times E(S) \rangle$ is the least group congruence on S.
- S is **proper** if

$$(a^+ = b^+ \text{ and } a\,\sigma\,b) \text{ implies } a = b;$$

this definition is left/right dual.

- Free inverse semigroups are proper.
- If S is proper, $S \to E(S) \times S/\sigma$ given by

$$s \mapsto (s^+, s\sigma)$$

is clearly a SET embedding.

The McAlister Theorems, 1974 Let S be an inverse semigroup.
(i) S is proper if and only if S is isomorphic to a P-**semigroup**;

(ii) S has a **proper cover**. That is, there exists a proper inverse semigroup $\widehat{S}$ and an idempotent separating morphism $\widehat{S} \twoheadrightarrow S$.

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Proper inverse semigroups

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2. The classical background
Proper inverse semigroups and generalisations

- Let S be Ehresmann; put $\sigma = \langle E \times E \rangle$.
- S/σ is **reduced**.
- A restriction semigroup S is **proper** if the following condition and its dual holds:
$$(a^+ = b^+ \text{ and } a \, \sigma \, b) \text{ implies } a = b.$$
- The free restriction semigroup is proper.
- Results for proper restriction semigroups involving semidirect products, analogous to those in the inverse case hold where **group** is replaced by **monoid** Branco, Cornock, El Qallali, Fountain, Gomes, G, Lawson, Szendrei; more recently, Kudryavtseva, Jones.
- The above has analogues in the one-sided case and ample case.

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2. The classical background
What makes such results involving semidirect products work?

Let M be a left Ehresmann monoid.

1. Suppose that $M = \langle X \rangle_{(2,1,0)}$. Put $T = \langle X \rangle_{(2,0)}$ *so that* T *is the* **monoid** *generated by* X.
2. $M = \langle T \cup E \rangle_{(2)}$ so that any $s \in M$ can be written as
$$s = t_0 e_1 t_1 \dots e_n t_n,$$
for some $t_0, \dots, t_n \in T$ and $e_1, \dots, e_n \in E$.
3. If the ample identities hold, e.g. in the inverse case or restriction case, then $s = f t_0 t_1 \dots t_n$ for some $f \in E$, so that $M = ET$.
4. The above is what is behind results connecting (left) restriction/ample/inverse monoids to semidirect products $Y \rtimes T$ of a semilattice Y and a monoid T.

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3. Some candidates for propriety: What do we know from former work?

Let M be left Ehresmann and let T be a submonoid. Then T acts on E by order-preserving maps via

$$t \cdot e = (te)^+.$$

If M is inverse/left ample/left restriction, then this action is by *morphisms* of the semilattice E.

3. Some candidates for propriety: What are we looking for?

The old notion of 'proper' is no good - it leads inexorably to a semidirect product construction, which is no longer appropriate.

- Want condition **P** for left Ehresmann monoids such that:
  (i) left Ehresmann monoids satisfying **P** have their structure described by monoids acting on semilattices;
  (ii) if M is left Ehresmann then there exists a left Ehresmann $\widehat{M}$ satisfying **P** and a projection-separating morphism

  $$\widehat{M} \twoheadrightarrow M,$$

  i.e. $\widehat{M}$ is a **cover** of M;
  (iii) free left Ehresmann monoids satisfy **P**.
  (iv) **P** plays a role in defining categories and varieties of left Ehresmann monoids.

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3. Some candidates for propriety: Generators and T-normal form Branco, Gomes, G

Let M be a left Ehresmann monoid.

Suppose that $M = \langle E \cup T \rangle_{(2)}$ where T is a submonoid of M.

Any $x \in M$ can be written as

$$x = t_0 e_1 t_1 \dots e_n t_n,$$

where $n \geq 0, e_1, ..., e_n \in E, t_1, ..., t_{n-1} \in T \setminus \{1\}, t_0, t_n \in T$ and for $1 \leq i \leq n$

$$e_i < (t_i e_{i+1} \dots t_n)^+.$$

Such an expression is in T-**normal form** and may be effectively calculated.

M has **uniqueness of T-normal forms** if every $x \in M$ has a unique such expression.

3. Some candidates for propriety: Generators and T-normal form Branco, Gomes, G

Let M be a left Ehresmann monoid.

Suppose that $M = \langle E \cup T \rangle_{(2)}$ where T is a submonoid of M.

Any $x \in M$ can be written as

$$x = t_0 e_1 t_1 \dots e_n t_n,$$

where $n \geq 0, e_1, ..., e_n \in E, t_1, ..., t_{n-1} \in T \setminus \{1\}, t_0, t_n \in T$ and for $1 \leq i \leq n$

$$e_i < (t_i e_{i+1} \dots t_n)^+.$$

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M is said to be **strongly T-proper** if for all $u, v \in T$,

$$u \,\sigma\, v \Rightarrow u = v;$$

M is said to be **very T-proper** if for all $u, v \in T$,

$$u \,\sigma\, v \Rightarrow u^+ v = v^+ u;$$

M is said to be **T-proper** if for all $u, v \in T, e, f \in E$

$$(ue)^+ = (ve)^+ \text{ and } ue \,\sigma\, ve, \text{ then } ue = ve.$$

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Proposition Let $M = \langle T \cup E \rangle_{(2)}$ be a left Ehresmann monoid. Then we have the following implications

$$\begin{aligned} M \text{ has uniqueness of } T\text{-normal forms} &\Rightarrow M \text{ is strongly } T\text{-proper} \\ &\Rightarrow M \text{ is very } T\text{-proper} \\ &\Rightarrow M \text{ is } T\text{-proper.} \end{aligned}$$

A **typical calculation** if $u^+v = v^+u$ and $(ue)^+ = (ve)^+$ then

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Theorem Branco, Gomes, G Let T be a monoid acting on the left of a semilattice E with identity, via order-preserving maps. Then there is a left Ehresmann monoid $\mathcal{P}_\ell(T,E)$ such that:

- $\mathcal{P}_\ell(T,E) = \langle T \cup E \rangle_{(2)}$;
- $\mathcal{P}_\ell(T,E)$ has uniqueness of T-normal forms;
- $\mathcal{P}_\ell(T,E)/\sigma \cong T$;
- the free left Ehresmann monoid on X is of the form $\mathcal{P}_\ell(X^*,E)$.

Theorem Branco, Gomes, G Let $M = \langle T \cup E \rangle_{(2)}$ be left Ehresmann, where T is a submonoid of M. Then $\mathcal{P}_\ell(T,E)$ is a cover for M.

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4. S-labelled trees: G, Hartmann and Wang
A category of left Ehresmann monoids

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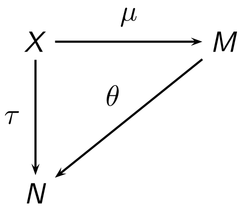
Let $X \neq \emptyset$ and let $\mathcal{C}(X)$ be the category such that

(i) **objects** are triples (M, X, μ) where M is left Ehresmann and $\mu : X \to M$ is a map such that $M = \langle X\mu \rangle_{(2,1,0)}$;

(ii) an **arrow** $\theta : (M, X, \mu) \to (N, X, \tau)$ is a morphism $\theta : M \to N$ such that $\tau = \mu\theta$.

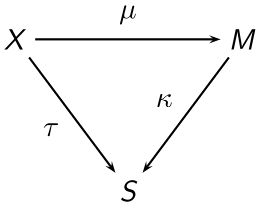


Then $\mathcal{C}(X)$ is the **category of** X**-generated left Ehresmann monoids**.

4. S-labelled trees: A category of left Ehresmann monoids

Let $X \neq \emptyset$, let S be a monoid let $\tau : X \to S$ such that $S = \langle X\tau \rangle$.

Let $\mathcal{C}(X, \tau, S)$ be the full subcategory of $\mathcal{C}(X)$ such that an object (M, X, μ) of $\mathcal{C}(X)$ lies in the subcategory if there is a morphism $\kappa : M \to S$ such that $\operatorname{Ker} \kappa = \sigma$ and $\mu\kappa = \tau$, and M is strongly T-proper, where T is the monoid $\langle X\mu \rangle$:



4. S-labelled trees: A category of left Ehresmann monoids

Let $F(X)$ be the free left Ehresmann monoid on X, with $\iota : X \to F(X)$.

S is a monoid, $\tau : X \to S$ such that $S = \langle X\tau \rangle$.

Theorem The category $\mathcal{C}(X, \tau, S)$ has initial object

$$(F(X)/\rho, X, \iota\rho^{\natural})$$

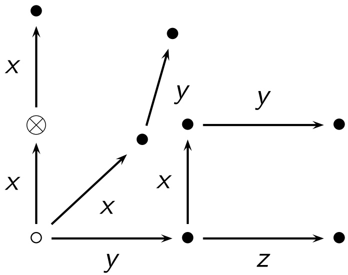
where

$$\rho = \langle (u\iota, v\iota) : u, v \in X^*, u\tau = v\tau \rangle.$$

Theorem The left Ehresmann monoid $F(X)/\rho$ is isomorphic to $\mathcal{P}_\ell(E, S)$ and hence has uniqueness of S-normal forms.

4. S-labelled trees: Free left Ehresmann monoid $F(X)$ Kambites 2011

X-labelled trees with root 'start' vertex and an 'end' vertex



Tree Γ: word $(y(xy)^+z^+)^+(xy)^+xx^+$

$\Gamma\Delta$: glue end of Γ to start of Δ

for $^+$ take $\otimes$ to $\circ$

Take equivalence classes under $\sim$, where $\Gamma \sim \Delta$ if Γ, Δ have a common retract

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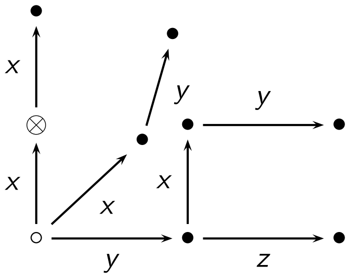

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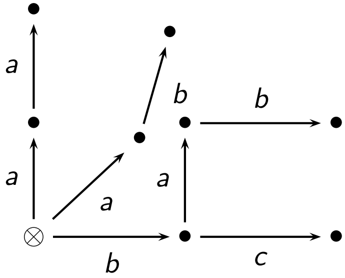
4. S-labelled trees: G, Hartmann and Wang

Relabel edges by elements of S: here $a = x\tau, b = y\tau, c = z\tau$

Delete vertices of degree 2



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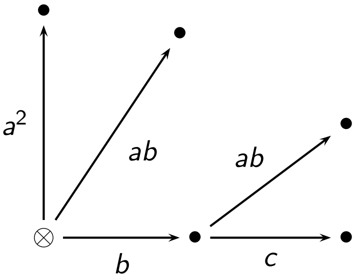

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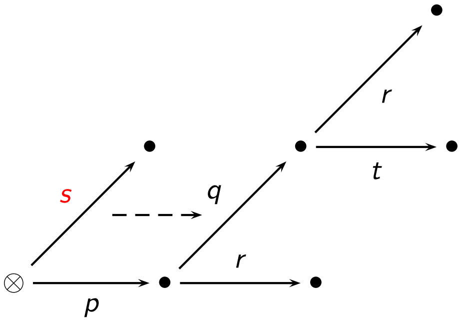

If $pq = sk$, for some k, 'fold' the branch labelled s to the path labelled pq:

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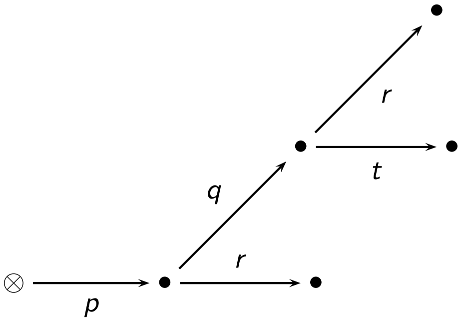

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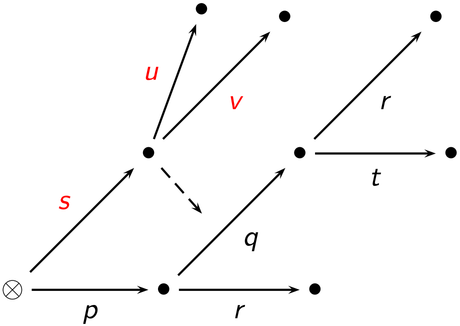

$pq = sk$; 'fold' the branch labelled s to the path labelled pq
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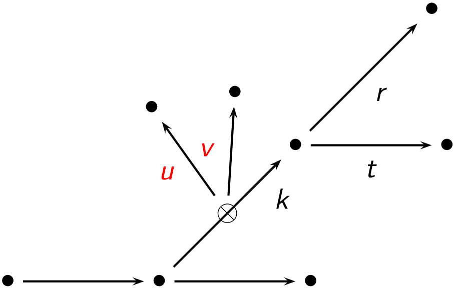

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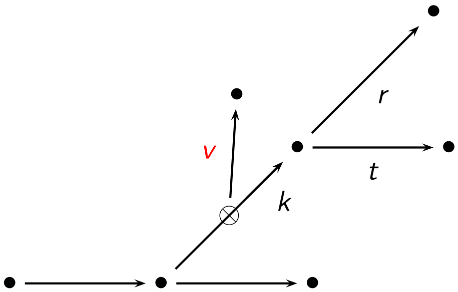

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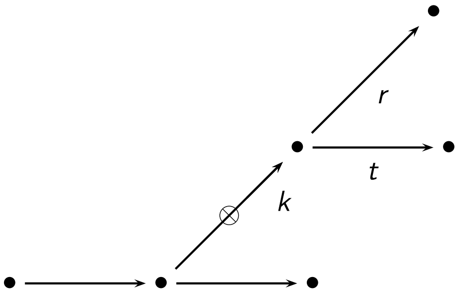

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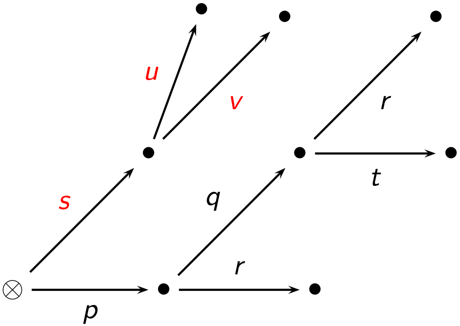

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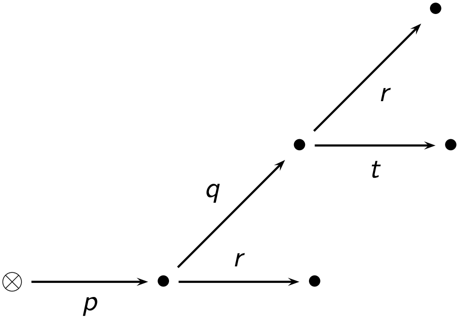

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S-labelled trees

Theorem G, Hartmann, Wang Let Σ, Δ be idempotent X-trees and let Σ_S, Δ_S be the corresponding S-trees. Then $\Sigma_S = \Delta_S$ in $F(X)/\rho$ if and only if Σ_S folds to Δ_S and vice versa.

Consequently as $F(X)/\rho$ has uniqueness of S-normal forms, and we have an effective procedure to obtain such, the word problem in $F(X)/\rho$ is solvable (modulo solving systems of equations in S).

Questions:

1. The word problem in the corresponding very T-proper case.
2. Are the subalgebras of $\mathcal{P}_\ell(T, E)$ exactly those satisfying some properness condition?
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