Monoids Acting by Isometric Embeddings

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Question

How much geometry is there in a finitely generated monoid?

Outline

- *Geometric group theory*
- *Geometric semigroup theory and semimetric spaces*
- *Monoids acting on semimetric spaces*
- *Groups acting on semimetric spaces*

Geometric Group Theory (Take 1)

Idea

Diagrams *are very useful when reasoning with discrete groups*

Examples

- Cayley graphs
- Schreier graphs
- van Kampen diagrams
- Automata
- ...

Geometric Group Theory (Take 2)

Idea

Groups have a natural metric structure, an understanding of which is vital to understanding their algebraic structure.

Fundamental Observation (Švarc, Milnor)

A discrete group acting in a suitably controlled way on a metric space **resembles** *that space.*

Conclusion

Discrete groups can be studied through their actions on metric spaces.

Groups as metric spaces

Let G be a group generated by a **symmetric** subset A.

Definition

The **distance** $d(g, h)$ from $g \in G$ to $h \in G$ is the shortest length of a sequence $a_1, \ldots a_n \in A$ such that

$$ga_1 a_2 \ldots a_n = h.$$

Properties

- *Distance is* **symmetric** *because A is symmetric.*
- *Distance is* **everywhere defined** *because G has no right ideals.*

Theorem (The Švarc-Milnor Lemma)

Let G be a group acting properly and cocompactly by isometries on a proper geodesic metric space X. Then G is finitely generated and **quasi-isometric** *to X.*

Definition

Metric spaces are X and Y are **quasi-isometric** if there is a function $f : X \to Y$ and a constant λ such that

(i) for all $x, y \in X$,

$$\frac{1}{\lambda} d_X(x, y) - \lambda \ \leq \ d_Y(f(x), f(y)) \ \leq \ \lambda d_X(x, y) + \lambda$$

(ii) every point in Y is within distance λ of a point in $f(X)$.

Proposition

The quasi-isometry class of a finitely generated group is independent of the chosen finite generating set.

Definition

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(ii) every point in Y is within distance λ of a point in $f(X)$.

Proposition

Quasi-isometry is an equivalence relation on metric spaces.

Idea

The quasi-isometry class of a space captures the information which remains when the space is viewed "from far away".

Geometric Semigroup Theory (Take 1)

Idea

Diagrams *are very useful when reasoning with semigroups.*

Examples

- Cayley graphs
- Schützenberger graphs (later)
- van Kampen diagrams (Remmers)
- Munn trees
- Eggbox diagrams
- Automata
- ...

Geometric Semigroup Theory (Take 2)

Question

Do semigroups have a natural "metric" structure?

Observations

Distance in a semigroup can be defined as for a group **but** *it is*

- **not symmetric** *(no symmetric generating sets);*
- **not everywhere defined** *(right ideals).*

Geometric Semigroup Theory (Take 3)

Definition

A **semimetric space** is a set X equipped with a function

$$d : X \times X \to \{r \in \mathbb{R} \mid r \geq 0\} \cup \{\infty\}$$

such that for all $x, y, z \in X$:

- $d(x, y) = 0 \iff x = y$;
- $d(x, z) \leq d(x, y) + d(y, z)$.

Notation

$\mathbb{R}^\infty = \{r \in \mathbb{R} \mid r \geq 0\} \cup \{\infty\}$ *(with the obvious order, + and ×).*

Examples of Semimetric Spaces

Example (The Directed Line (Take 1))

Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^\infty$ by

$$d(x,y) = \begin{cases} y - x & \text{if } x \leq y \\ \infty & \text{otherwise.} \end{cases}$$

Example (The Directed Line (Take 2))

Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^\infty$ by

$$d(x,y) = \begin{cases} |y - x| & \text{if } x = y \text{ or } x < \lceil y \rceil \\ \infty & \text{otherwise.} \end{cases}$$

(The restriction to $[0,1]$ is the **directed unit interval**.)

Examples of Semimetric Spaces

Example (Directed Graphs)

Let X be a directed graph. Define the distance from vertex x to vertex y to be the minimal length of a directed path from x to y.

(This can be made into a **geodesic** semimetric space, by gluing in a copy of the directed unit interval for each edge.)

Quasi-isometries of Semimetric Spaces

Definition

A function $f : X \to Y$ between semimetric spaces is called a **quasi-isometry** if there is a constant $\lambda < \infty$ such that

$$\frac{1}{\lambda} d_X(x_1, x_2) - \lambda \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \lambda$$

and for every point $y \in Y$ there is a point $x \in X$ such that

$$d(f(x), y) \leq \lambda \text{ and } d(y, f(x)) \leq \lambda.$$

Proposition

Quasi-isometry is an equivalence relation on semimetric spaces.

Quasi-isometries of Semigroups

Proposition

Let A and B be finite generating sets for a semigroup S. Then the corresponding semimetric spaces are quasi-isometric.

Definition

Two finitely generated semigroups are **quasi-isometric** if they are quasi-isometric with respect to any/all finite generating sets.

Fact

Many important properties of groups are **quasi-isometry invariant**.

Question

What about properties of semigroups?

Groups as Semimetric Spaces

Fact

Metric spaces are also semimetric.

Question

Are groups metric?

Answer

No! Not with respect to monoid generating sets . . .

Quasimetric Spaces

Definition

A semimetric space X is called **strongly connected** if $d(x, y) \neq \infty$ for all $x, y \in X$.

Definition

A semimetric space X is **quasimetric** if it is strongly connected and there is a constant $\lambda < \infty$ such that

$$d(x, y) \leq \lambda d(y, x) + \lambda$$

for all $x, y \in X$.

Proposition

A space X is quasi-metric $\iff$ it is quasi-isometric to a metric space.

Fact

A group G with a monoid generating set is a quasimetric space.

Balls in Semimetric Spaces

Definition

Let X be a semimetric space, $x \in X$ and $\epsilon \in \mathbb{R}^{\infty}$.

- The **out-ball** of radius ϵ around x is

$$\overrightarrow{\mathcal{B}}_{\epsilon}(x) = \{y \in X \mid d(x, y) \leq \epsilon\}.$$

- The **in-ball** of radius ϵ around x is

$$\overleftarrow{\mathcal{B}}_{\epsilon}(x) = \{y \in X \mid d(y, x) \leq \epsilon\}.$$

- The **strong-ball** of radius ϵ around x is

$$\mathcal{B}_{\epsilon}(x) = \overrightarrow{\mathcal{B}}_{\epsilon}(x) \cap \overleftarrow{\mathcal{B}}_{\epsilon}(x).$$

Actions on Semimetric Spaces

Let M be a monoid acting by isometric embeddings on a semimetric space X.

Definition

The action is called **cobounded** if there exists a strong ball $B = \mathcal{B}_\epsilon(x)$ of finite radius such that

$$X = \bigcup_{m \in M} mB.$$

Definition

The action is called **outward proper** if for every out-ball $B = \overrightarrow{\mathcal{B}}_\epsilon(x)$ of finite radius the set

$$\{m \in M \mid B \cap mB \neq \emptyset\}$$

is finite.

Actions and Ideals

Definition

A point x_0 in a semimetric space X is called a **basepoint** if for every $x \in X$ we have $d(x_0, x) < \infty$.

M a monoid acting by isometric embeddings on a semimetric space X.

Definition

The action is called **idealistic** at a basepoint $x_0 \in X$ if

$$d(mx_0, nx_0) < \infty \implies nM \subseteq mM.$$

for all $m, n \in M$.

Remark

If M is a **group** *then the action is idealistic exactly if X has a basepoint.*

Remark

If M acts idealistically on a **strongly connected** *space then M is a group.*

Theorem (Švarc-Milnor Lemma for Isometric Embeddings)

Let M be a monoid acting outward properly, coboundedly and idealistically by isometric embeddings on a geodesic semimetric space X.

Then M is finitely generated and quasi-isometric to X.

Theorem (Švarc-Milnor for Groups Acting on Semimetric Spaces)

Let G be a group acting outward properly and coboundedly by isometries on a geodesic semimetric space X with basepoint.

Then G is finitely generated and quasi-isometric to X.

In particular, X is quasi-metric.

Proposition

Let M be a finitely generated cancellative monoid. Then M acts coboundedly, outward properly and idealistically on its Cayley graph.

Corollary

A cancellative monoid is finitely generated if and only if it acts coboundedly, outward properly and idealistically on a geodesic semimetric space.

Theorem

Let M be a left unitary submonoid of a finitely generated cancellative monoid N.

Suppose there is a finite set P of right units such that $MP = N$. ("finite index?!")

Then M is finitely generated and quasi-isometric to N.

Corollary

Let F be a finitely generated free monoid of rank k, and G a finite group of order n.

Then $F * G$ is quasi-isometric to a free monoid of rank kn.

Ideals and Green's Relations

We define a pre-order $\leq_{\mathcal{R}}$ on a monoid M by ...

- $x \leq_{\mathcal{R}} y \iff xM \subseteq yM$;

From this we obtain an equivalence relation ...

- $x \mathcal{R} y \iff xM = yM \iff x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$

Similarly ...

- $x \leq_{\mathcal{L}} y \iff Mx \subseteq My, \quad x \mathcal{L} y \iff Mx = My$
- $x \leq_{\mathcal{J}} y \iff MxM \subseteq MyM, \quad x \mathcal{J} y \iff MxM = My$

We also define equivalence relations ...

- $x \mathcal{H} y \iff x \mathcal{R} y$ and $x \mathcal{L} y$;
- $x \mathcal{D} y \iff x \mathcal{R} z$ and $z \mathcal{L} y$ for some $z \in M$;

These relations encapsulate the (left, right and two-sided) ideal structure of M and are fundamental to its structure.

Schützenberger Groups

Let H be an $\mathcal{H}$-class of a semigroup S.

Definition

The **Schützenberger group** of H is the group of all permutations of H which arise as restrictions of the left translation action of elements of S.

Fact

The Schützenberger group acts naturally on the $\mathcal{R}$-class containing H.

Fact

All Schützenberger groups in the same $\mathcal{D}$-class are isomorphic.

Fact

In a regular $\mathcal{D}$-class, they are isomorphic to the maximal subgroups.

Let S be a semigroup generated by a finite subset A.

Let R be an $\mathcal{R}$-class of S.

Definition

The **Schützenberger graph** of R is the directed graph with vertex set R, and an edge from $s \in R$ to $t \in R$ if there exists $x \in A$ such that $sx = t$.

(A maximal strongly connected component of the Cayley graph.)

Fact

The Schützenberger groups of $\mathcal{H}$-classes in $\mathcal{R}$ act naturally by isometries on the Schützenberger graph of R.

Theorem

Let G be a Schützenberger group of a finitely generated semigroup S, acting on the associated Schützenberger graph. The action is

- *outward proper and by isometries;*
- *cobounded $\iff$ the $\mathcal{R}$-class contains finitely many $\mathcal{H}$-classes.*

Corollary

An $\mathcal{R}$-class with finitely many $\mathcal{H}$-classes has Schützenberger groups quasi-isometric to its Schützenberger graph.

Corollary/Remark

Such Schützenberger graphs are quasi-metric. (So the previous corollary can also be obtained by symmetrizing.)

Finite Presentations (1)

Theorem

For finitely generated monoids with finitely many left and right ideals, finitely presentability is a quasi-isometry invariant.

Proof.

- If M and N are quasi-isometric, their Schützenberger **graphs** are quasi-isometric.
- So by the theorem, their Schützenberger **groups** are quasi-isometric.
- Finite presentability is a quasi-isometry invariant of groups.
- A finitely generated monoid with finitely many left and right ideals is finitely presented if and only if its Schützenberger groups are all finitely presented (Ruskuc).

□

Finite Presentations (2)

Theorem

For finitely generated monoids with finitely many left and right ideals, finitely presentability is a quasi-isometry invariant.

Theorem

For finitely generated left cancellative monoids, finite presentability is a quasi-isometry invariant.

Theorem

For finitely presentable left cancellative monoids, having solvable word problem is a quasi-isometry invariant.

Finite Presentations (3)

Question

Is finite presentability a quasi-isometry invariant of **arbitrary** *finitely generated monoids? Or even an isometry invariant?*

Theorem

There is an infinite family of 5*-generated monoids, all isometric, containing monoids with word problems of more or less arbitrary complexity and monoids which are not recursively presentable.*

Conjecture

Finite presentability is not even an isometry invariant of finitely generated monoids.

Growth

Definition

Let M be a monoid generated by a finite subset X. The **growth function** of M is the function

$$\mathbb{N} \to \mathbb{N}, \ n \mapsto |\{m \in M \mid d(1, m) \leq n\}| \, .$$

The **growth type** of M is the asymptotic growth class of the growth function.

Theorem

Growth type is a quasi-isometry invariant of monoids.

Ends of Monoids

Definition (Jackson and Kilibarda)

The **number of ends** of a monoid is the greatest number of infinite connected components which can be obtained by removing finitely many vertices from its Cayley graph.

Theorem

Number of ends is a quasi-isometry invariant of monoids.

Corollary (originally due to Jackson and Kilibarda)

The number of ends of a monoid is invariant under change of generators.

Question

What semigroup-theoretic constructions preserve quasi-isometry type?

Proposition (well-known)

Let G be a group and N a finite normal subgroup. Then G is quasi-isometric to G/N.

Proposition

Let S be a semigroup and σ a congruence with classes of bounded diameter. Then S is quasi-isometric to S/σ.

The Future

Question

What properties of monoids/semigroups are quasi-isometry invariant?

Question

What properties of monoids/semigroups are isometry invariant?

Question

Can we replace isometric embeddings with **contractions***?*

Question

Can we replace (directed) geometry with (directed) topology?

Question

Can we study relations, as well as generators?