Purity for S-acts

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What is this talk about? An old question for S-acts
What are S-acts?

Throughout, S is a **monoid**.

A **(right)** S**-act** is a set A together with a map

$$A \times S \to A, \ (a, s) \mapsto as$$

such that for all $a \in A, s, t \in S$

$$a1 = a \text{ and } (as)t = a(st).$$

Remark (i) For any $s \in S$, we have an operation $\rho_s : A \to A$ given by $a\rho_s = as$. The function $\rho : S \to \mathcal{T}_A$ given by

$$s\rho = \rho_s$$

is a monoid morphism.

(ii) Conversely, if $\theta : S \to \mathcal{T}_A$ is a morphism, define

$$as = a(s\theta),$$

then check that A is then an S-act.

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Examples of S-acts

1. S is an S-act
2. Any right ideal of S is an S-act.
3. Let ρ be a right congruence on S. Let

$$S/\rho = \{[a] : a \in S\}$$

and define $[a]s = [as]$. Then S/ρ is an S-act. For any $[a] \in S/\rho$ we have

$$[a] = [1]a.$$

Elementary observations for S-acts

- An S-**morphism** from A to B is a map $\alpha : A \to B$ with $(as)\alpha = (a\alpha)s$ for all $a \in A, s \in S$.
- S-acts and S-morphisms form a category - products are products, coproducts are disjoint unions
- We have usual definitions of **free, projective, injective**, etc. including variations on **flat**.
- Free S-acts are disjoint unions of copies of S.

Free S-acts

Let X be a set. By general nonsense, the free S-act $\mathcal{F}_S(X)$ on X exists.

Construction of $\mathcal{F}_S(X)$. Let

$$F_S(X) = X \times S$$

and define

$$(x, s)t = (x, st).$$

Then it is easy to check that $F_S(X)$ is an S-act. With $x \mapsto (x, 1)$, we have $F_S(X)$ is free on X.
Notice that

$$(x, s) = (x, 1)s \equiv xs.$$

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Congruences for S-acts

- A **congruence** ρ on A is an equivalence relation such that

$$a \,\rho\, b \Rightarrow as \,\rho\, bs$$

  for all $a, b \in A$ and $s \in S$.
- ρ is **finitely generated** if ρ is the smallest congruence containing a finite set $H \subseteq A \times A$.
- If ρ is a congruence on A then A/ρ is an S-act; all monogenic S-acts are of the form S/ρ.
- For $a \in A$ write aS for $\{as : s \in S\}$; then aS is an S-subact of A.
- If $X = \{x_1, \ldots, x_n\}$, then

$$F_S(n) = F_S(X) = (x_1, 1)S \cup \ldots \cup (x_n, 1)S$$

  and we often write

$$F_S(n) = x_1 S \cup \ldots \cup x_n S.$$

Finite generation and finite presentation

A is **finitely generated** if

$$A = a_1 S \cup \ldots \cup a_n S$$

for some $a_i \in A$. and **finitely presented** if

$$A \cong F_S(n)/\rho$$

for some finitely generated free $F_S(n)$ and finitely generated congruence ρ.

Algebraically closed S-acts

Let A be an S-act. An **equation** over A has the form

$$xs = xt, \; xs = yt \text{ or } xs = a$$

where x, y are variables, $s, t \in S$ and $a \in A$.

A set of equations and inequations is **consistent** if it has a solution in some S-act $B \supseteq A$.

Definition A is **algebraically closed** or **absolutely pure** if every finite consistent set of equations over A has a solution in A.

Definition A is **almost pure** if every finite consistent set of equations *in one variable* over A has a solution in A.

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Algebraic closure and injectivity

An S-act T is **injective** if for any S-acts A, B and S-morphisms

$$\phi : A \to B, \psi : A \to T$$

with ϕ one-one, there exists an S-morphism $\theta : B \to T$ such that

$$\phi\theta = \psi.$$

Proposition (G, 19[redacted]) An S-act T is injective if and only if every consistent system of equations over T has a solution in T.

Restrictions on the A, B give restricted notions of injectivity and these are related to solutions of special consistent systems of equations.

The Baer criterion and injectivity

Theorem The following are equivalent for an S-act A:

1. every finite consistent system of equations over A of the form
$$xs_1 = a_1, \ldots, xs_n = a_n$$
has a solution in A;
2. for any finitely generated right ideal I of S and S-morphism $\psi : I \to A$, there exists an S-morphism $\theta : S \to A$ such that
$$\iota\theta = \psi;$$
3. for any finitely generated right ideal I of S and S-morphism $\psi : I \to A$, there exists an $a \in A$ such that $s\psi = as$ for all $s \in I$.

(The last criterion is the **Baer criterion**).

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Algebraic closure and injectivity

Recall and S-act C is **absolutely pure** if every finite consistent system of equations over C, has a solution in C.

Proposition An S-act C is algebraically closed if and only if for any S-acts A, B and S-morphisms

$$\phi : A \to B, \psi : A \to T$$

with ϕ one-one, B finitely presented and A finitely generated, there exists an S-morphism $\theta : B \to C$ such that

$$\phi\theta = \psi.$$

Absolute purity vs almost purity

Definition A monoid S is **completely right pure** if all S-acts are absolutely pure.

Theorem (G) Suppose that all S-acts are almost pure. Then S is completely right pure.

My question

Does there exist an almost pure S-act that is not absolutely pure?????

Completely (right) injective monoids

Definition A monoid S is **completely right injective** if all S-acts are injective.

Theorem (Fountain, 1974) A monoid S is completely right injective if and only if S has a left zero and S satisfies
(*) for any right ideal I of S and right congruence ρ on S, there is an $s \in I$ such that for all $u, v \in S, w \in I$, $sw \, \rho \, w$ and if $u \, \rho \, v$ then $su \, \rho \, sv$.

Theorem (Gould, 1991) A monoid S is completely right pure if and only if S has a local zeros and S satisfies
(**) for any finitely generated right ideal I of S and finitely generated right congruence ρ on S, there is an $s \in I$ such that for all $u, v \in S, w \in I$, $sw \, \rho \, w$ and if $u \, \rho \, v$ then $su \, \rho \, sv$.