An introduction to Zappa-Szép Products

Rida-e-Zenab

History of Zappa-Szép products

- Zappa-Szép product was developed by *G Zappa* in 1940. He is an atalian mathematician and a famous group theorist.
- In 1941 Casadio and in 1950 Redei studied the variations and generalizations of Zappa-Szép products in setting of groups.
- Redei, Szép and Tibiletti used these products to discover properties of groups in numerous papers. Szép introduced the relations of this product and used them to study structural properties of groups e.g., normal subgroups in 1950. He also initiated the study of similar products in setting other than groups in 1958 and 1968.
- In 1950 Tolo studied factorizable semigroups which are internal Zappa-Szép products if factorizations are unique.
- Zappa-Szép products in semigroup theory were introduced by M. Kunze in 1983.

History of Zappa-Szép products

- Zappa-Szép product was developed by *G Zappa* in 1940. He is an atalian mathematician and a famous group theorist.

- In 1941 Casadio and in 1950 Redei studied the variations and generalizations of Zappa-Szép products in setting of groups.

- Redei, Szép and Tibiletti used these products to discover properties of groups in numerous papers. Szép introduced the relations of this

  product and used them to study structural properties of groups e.g., normal subgroups in 1950. He also initiated the study of similar products in setting other than groups in 1958 and 1968.

- In 1950 Tolo studied factorizable semigroups which are internal Zappa-Szép products if factorizations are unique.

- Zappa-Szép products in semigroup theory were introduced by M. Kunze in 1983.

History of Zappa-Szép products

- Zappa-Szép product was developed by *G Zappa* in 1940. He is an atalian mathematician and a famous group theorist.
- In 1941 Casadio and in 1950 Redei studied the variations and generalizations of Zappa-Szép products in setting of groups.
- Redei, Szép and Tibiletti used these products to discover properties of groups in numerous papers. Szép introduced the relations of this

  product and used them to study structural properties of groups e.g., normal subgroups in 1950. He also initiated the study of similar products in setting other than groups in 1958 and 1968.
- In 1950 Tolo studied factorizable semigroups which are internal Zappa-Szép products if factorizations are unique.
- Zappa-Szép products in semigroup theory were introduced by M. Kunze in 1983.

History of Zappa-Szép products

- Zappa-Szép product was developed by *G Zappa* in 1940. He is an atalian mathematician and a famous group theorist.
- In 1941 Casadio and in 1950 Redei studied the variations and generalizations of Zappa-Szép products in setting of groups.
- Redei, Szép and Tibiletti used these products to discover properties of groups in numerous papers. Szép introduced the relations of this

  product and used them to study structural properties of groups e.g., normal subgroups in 1950. He also initiated the study of similar products in setting other than groups in 1958 and 1968.
- In 1950 Tolo studied factorizable semigroups which are internal Zappa-Szép products if factorizations are unique.
- Zappa-Szép products in semigroup theory were introduced by M. Kunze in 1983.

History of Zappa-Szép products

- Zappa-Szép product was developed by *G Zappa* in 1940. He is an atalian mathematician and a famous group theorist.
- In 1941 Casadio and in 1950 Redei studied the variations and generalizations of Zappa-Szép products in setting of groups.
- Redei, Szép and Tibiletti used these products to discover properties of groups in numerous papers. Szép introduced the relations of this

  product and used them to study structural properties of groups e.g., normal subgroups in 1950. He also initiated the study of similar products in setting other than groups in 1958 and 1968.
- In 1950 Tolo studied factorizable semigroups which are internal Zappa-Szép products if factorizations are unique.
- Zappa-Szép products in semigroup theory were introduced by M. Kunze in 1983.

History of Zappa-Szép products

- Zappa-Szép product was developed by *G Zappa* in 1940. He is an atalian mathematician and a famous group theorist.
- In 1941 Casadio and in 1950 Redei studied the variations and generalizations of Zappa-Szép products in setting of groups.
- Redei, Szép and Tibiletti used these products to discover properties of groups in numerous papers. Szép introduced the relations of this

  product and used them to study structural properties of groups e.g., normal subgroups in 1950. He also initiated the study of similar products in setting other than groups in 1958 and 1968.
- In 1950 Tolo studied factorizable semigroups which are internal Zappa-Szép products if factorizations are unique.
- Zappa-Szép products in semigroup theory were introduced by M. Kunze in 1983.

History of Zappa-Szép products

- In 1998 Lavers find conditions under which Zappa-Szép product of two finitely presented monoids is itself finitely presented.
- Brin extended the capability of Zappa-Szép products in categories and monoids in 2005.
- In 2007 M. Lawson studied Zappa-Szép product of a free monoid and a group from the view of self similar group action and completely determined their structure.
- Recently Suha wazan has studied Zappa-Szép products from the view of regular and inverse semigroups. She generalized some results of Lawson and find necessary and sufficient conditions for these products.

History of Zappa-Szép products

- In 1998 Lavers find conditions under which Zappa-Szép product of two finitely presented monoids is itself finitely presented.
- Brin extended the capability of Zappa-Szép products in categories and monoids in 2005.
- In 2007 M. Lawson studied Zappa-Szép product of a free monoid and a group from the view of self similar group action and completely determined their structure.
- Recently Suha wazan has studied Zappa-Szép products from the view of regular and inverse semigroups. She generalized some results of Lawson and find necessary and sufficient conditions for these products.

History of Zappa-Szép products

- In 1998 Lavers find conditions under which Zappa-Szép product of two finitely presented monoids is itself finitely presented.
- Brin extended the capability of Zappa-Szép products in categories and monoids in 2005.
- In 2007 M. Lawson studied Zappa-Szép product of a free monoid and a group from the view of self similar group action and completely determined their structure.
- Recently Suha wazan has studied Zappa-Szép products from the view of regular and inverse semigroups. She generalized some results of Lawson and find necessary and sufficient conditions for these products.

History of Zappa-Szép products

- In 1998 Lavers find conditions under which Zappa-Szép product of two finitely presented monoids is itself finitely presented.
- Brin extended the capability of Zappa-Szép products in categories and monoids in 2005.
- In 2007 M. Lawson studied Zappa-Szép product of a free monoid and a group from the view of self similar group action and completely determined their structure.
- Recently Suha wazan has studied Zappa-Szép products from the view of regular and inverse semigroups. She generalized some results of Lawson and find necessary and sufficient conditions for these products.

Definition

- Let G be a group with identity element e. Let N be normal subgroup of G and H be a subgroup of G. Then G is said to be a *semidirect product* of N and H if the following hold.
  1. $G = HN$ and
  2. $N \cap H = \{e\}$.

  It is denoted by
  $$G = N \rtimes H.$$

Internal Zappa-Szép product

Definition

Let G be a group with identity element e., and let H and K be subgroups of G. THFAE

- $G = HK$ and $H \cap K = \{e\}$
- For each $g \in G$ there exists a unique $h \in H$ and unique $k \in K$ such that $g = hk$.

If either (and hence both) of these statements hold, then G is said to be internal Zappa-Szép product of H and K and is denoted by $G = H \bowtie K$.

Internal Zappa-Szép product

Definition

Let G be a group with identity element e., and let H and K be subgroups of G. THFAE

- $G = HK$ and $H \cap K = \{e\}$
- For each $g \in G$ there exists a unique $h \in H$ and unique $k \in K$ such that $g = hk$.

If either (and hence both) of these statements hold, then G is said to be internal Zappa-Szép product of H and K and is denoted by $G = H \bowtie K$.

Internal Zappa-Szép product

Definition

Let G be a group with identity element e., and let H and K be subgroups of G. THFAE

- $G = HK$ and $H \cap K = \{e\}$
- For each $g \in G$ there exists a unique $h \in H$ and unique $k \in K$ such that $g = hk$.

If either (and hence both) of these statements hold, then G is said to be internal Zappa-Szép product of H and K and is denoted by $G = H \bowtie K$.

External Zappa-Szép product

Definition

Let H and K be groups (and let e denote each group's identity element) and suppose we have bijective maps

$$\begin{aligned} H \times K \to H, &\quad (h,k) \mapsto k \cdot h \\ H \times K \to K, &\quad (h,k) \mapsto k^h \end{aligned}$$

such that for all $h, h' \in H, k, k' \in K$,
(ZS1) $kk' \cdot h = k \cdot (k' \cdot h)$;
(ZS2) $k \cdot (hh') = (k \cdot h)(k^h \cdot h')$;
(ZS3) $(k^h)^{h'} = k^{hh'}$;
(ZS4) $(kk')^h = k^{k' \cdot h} k'^h$.
(ZS5) $k \cdot e_H = e_H$
(ZS6) $k^{e_H} = k$
(ZS7) $e_K.h = h$
(ZS8) $e_K^h = e_K$

External Zappa-Szép product

On the cartesian product $H \times K$, define a multiplication and an inversion map respectively,

- $$(h,k)(h',k') = (h(k \cdot h'), k^{h'}k').$$

- $$(h,k)^{-1} = (k^{-1} \cdot h^{-1}, (k^{-1})^{h^{-1}}).$$

Then $H \times K$ is a group, called *external Zappa-Szép product* of groups H and K and denoted by $H \bowtie K$.
The subsets $H \times \{e\}$ and $\{e\} \times K$ are subgroups isomorphic to H and K, respectively, and $H \bowtie K$ is, in fact, an internal Zappa-Szép product of $H \times \{e\}$ and $\{e\} \times K$.

External Zappa-Szép product

On the cartesian product $H \times K$, define a multiplication and an inversion map respectively,

- $$(h, k)(h', k') = (h(k \cdot h'), k^{h'} k').$$
- $$(h, k)^{-1} = (k^{-1} \cdot h^{-1}, (k^{-1})^{h^{-1}}).$$

Then $H \times K$ is a group, called *external Zappa-Szép product* of groups H and K and denoted by $H \bowtie K$.
The subsets $H \times \{e\}$ and $\{e\} \times K$ are subgroups isomorphic to H and K, respectively, and $H \bowtie K$ is, in fact, an internal Zappa-Szép product of $H \times \{e\}$ and $\{e\} \times K$.

Examples

Example 1

Let $G = GL(n, C)$, the general linear group of invertible $n \times n$ matrices over the field of complex numbers. For each matrix A in G, the QR decomposition asserts that there exists a unique unitary matrix Q and a unique upper triangular matrix R with positive real entries on the main diagonal such that $A = QR$. Thus G is a Zappa-Szép product of the unitary group $U(n)$ and the group (say) K of upper triangular matrices with positive diagonal entries.

Example 2

One of the most important examples of Zappa-Szép product is Hall's 1937 theorem on the existence of Sylow systems for soluble groups. This shows that every soluble group is a Zappa-Szép product of a Hall p-subgroup and a Sylow p-subgroup, and in fact that the group is a (multiple factor) Zappa Szép product of a certain set of representatives of its Sylow subgroups.

The concept of Zappa-Szép product in semigroups was introduced by *M. Kunze* in 1983. He investigated their role in transformation monoids and automata theory. He gave applications of Zappa-Szép product to translational hulls, Bruck- Reilly extensions and Rees matrix semigroups. In 1992 Kunze used the terminology Bilateral semigroup for Zappa-Szép product and studied aperiodic transformation semigroups (X, S) and investigated a strong decomposition of its elements out of idempotents.

External and internal Zappa-Szép product in semigroups

- Let S and T be semigroups and suppose we are given functions

$$S \times T \to S, \, (s,t) \mapsto t \cdot s \in S \dashrightarrow (1) \text{ and}$$

$$S \times T \to T, \, (s,t) \mapsto t^s \in T \dashrightarrow (2)$$

  where $s \in S$ and $t \in T$, satisfying the Zappa-Szép rules (ZS1), (ZS2), (ZS3) and (ZS4) developed by G. Zappa. Then the set $S \times T$ with the product defined by:

$$(s,t)(s'.t') = (s(t.s'), t^{s'}t')$$

  is a semigroup, called external Zappa-Szép product of S and T, which is written as $S \bowtie T$.

- A semigroup S is called internal Zappa-Szép product of subsemirоups A and B, if each $s \in S$ is uniquely expressible as $s = ab$.

External and internal Zappa-Szép product in semigroups

- Let S and T be semigroups and suppose we are given functions

$$S \times T \to S,\ (s,t) \mapsto t \cdot s \in S \dashrightarrow (1) \text{ and}$$

$$S \times T \to T,\ (s,t) \mapsto t^s \in T \dashrightarrow (2)$$

  where $s \in S$ and $t \in T$, satisfying the Zappa-Szép rules (ZS1), (ZS2), (ZS3) and (ZS4) developed by G. Zappa. Then the set $S \times T$ with the product defined by:

$$(s,t)(s'.t') = (s(t.s'), t^{s'}t')$$

  is a semigroup, called external Zappa-Szép product of S and T, which is written as $S \bowtie T$.

- A semigroup S is called internal Zappa-Szép product of subsemiroups A and B, if each $s \in S$ is uniquely expressible as $s = ab$.

External and internal Zappa-Szép product of monoids

Definition

If S and T are monoids with identities 1_S and 1_T respectively, then external and internal Zappa-Szép products of S and T are defined as above for semigroups satisfying axioms (ZSI) to (ZS8) developed by G Zappa.

$S \times T$ is a monoid with identity $(1_S, 1_T) \in S \times T$.

Theorem

$M = S \bowtie T$ is the external Zappa-Szép product of monoids S and T if and only if $M = \bar{S}\bar{T}$ is the internal Zappa-Szép product of submonoids $\bar{S}$ and $\bar{T}$ isomorphic to S and T respectively.

External and internal Zappa-Szép product of monoids

Definition

If S and T are monoids with identities 1_S and 1_T respectively, then external and internal Zappa-Szép products of S and T are defined as above for semigroups satisfying axioms (ZSI) to (ZS8) developed by G Zappa.

$S \times T$ is a monoid with identity $(1_S, 1_T) \in S \times T$.

Theorem

$M = S \bowtie T$ is the external Zappa-Szép product of monoids S and T if and only if $M = \bar{S}\,\bar{T}$ is the internal Zappa-Szép product of submonoids $\bar{S}$ and $\bar{T}$ isomorphic to S and T respectively.

Properties of Zappa-Szép product of monoids

Theorem

Let $M = S \bowtie T$ be a Zappa-Szép product of S and T. Then for $s_1, s_2 \in S, t_1, t_2 \in T$

- $(s_1, t_1)\ \mathcal{R}\ (s_2, t_2) \Longrightarrow s_1\ \mathcal{R}\ s_2$ in S.
- $(s_1, t_1)\ \mathcal{L}\ (s_2, t_2) \Longrightarrow t_1\ \mathcal{L}\ t_2$ in T.
- $(s_1, t_1) \leq_{\mathcal{R}} (s_2, t_2) \Longrightarrow s_1 \leq_{\mathcal{R}} s_2$ in S.
- $(s_1, t_1) \leq_{\mathcal{L}} (s_2, t_2) \Longrightarrow t_1 \leq_{\mathcal{L}} t_2$ in T.

Properties of Zappa-Szép product of monoids

Theorem

Let $M = S \bowtie T$ be a Zappa-Szép product of S and T. Then for $s_1, s_2 \in S, t_1, t_2 \in T$

- $(s_1, t_1)\ \mathcal{R}\ (s_2, t_2) \Longrightarrow s_1\ \mathcal{R}\ s_2$ in S.
- $(s_1, t_1)\ \mathcal{L}\ (s_2, t_2) \Longrightarrow t_1\ \mathcal{L}\ t_2$ in T.
- $(s_1, t_1) \leq_{\mathcal{R}} (s_2, t_2) \Longrightarrow s_1 \leq_{\mathcal{R}} s_2$ in S.
- $(s_1, t_1) \leq_{\mathcal{L}} (s_2, t_2) \Longrightarrow t_1 \leq_{\mathcal{L}} t_2$ in T.

Properties of Zappa-Szép product of monoids

Theorem

Let $M = S \bowtie T$ be a Zappa-Szép product of S and T. Then for $s_1, s_2 \in S, t_1, t_2 \in T$

- $(s_1, t_1) \ \mathcal{R} \ (s_2, t_2) \Longrightarrow s_1 \ \mathcal{R} \ s_2$ in S.
- $(s_1, t_1) \ \mathcal{L} \ (s_2, t_2) \Longrightarrow t_1 \ \mathcal{L} \ t_2$ in T.
- $(s_1, t_1) \leqq_{\mathcal{R}} (s_2, t_2) \Longrightarrow s_1 \leqq_{\mathcal{R}} s_2$ in S.
- $(s_1, t_1) \leqq_{\mathcal{L}} (s_2, t_2) \Longrightarrow t_1 \leqq_{\mathcal{L}} t_2$ in T.

Theorem

Let $M = S \bowtie T$ be a Zappa-Szép product of S and T. Then for $s_1, s_2 \in S, t_1, t_2 \in T$

- $(s_1, t_1)\ \mathcal{R}\ (s_2, t_2) \Longrightarrow s_1\ \mathcal{R}\ s_2$ in S.
- $(s_1, t_1)\ \mathcal{L}\ (s_2, t_2) \Longrightarrow t_1\ \mathcal{L}\ t_2$ in T.
- $(s_1, t_1) \leqq_{\mathcal{R}} (s_2, t_2) \Longrightarrow s_1 \leqq_{\mathcal{R}} s_2$ in S.
- $(s_1, t_1) \leqq_{\mathcal{L}} (s_2, t_2) \Longrightarrow t_1 \leqq_{\mathcal{L}} t_2$ in T.

Applications of Zappa-Szép product to semigroups

Transformation monoids

Let S be a transformation monoid acting on a set T where T is a right zero semigroup. Defining action of S on T by $(s,t) \to t^s = ts \in T$ and the action of T on S by $(s,t) \to t \cdot s = 1_S \in S$, we get the external Zappa-Szép product $M = S \bowtie T$ with binary operation

$$(s,t)(s'.t') = (s,t')$$

On the other hand, if we consider T as a left zero semigroup and define $(s,t) \to t \cdot s = s \in S$ and $(s,t) \to t^s = ts \in T$ we obtain a Zappa-Szép product with multiplication

$$(s,t)(s'.t') = (ss',ts').$$

The Bruck-Reilly extension of a monoid

Let S be a monoid and θ be an endomorphism of it. The Bruck-Reilly extension $BR(S,\theta)$ consists of triples (k,s,m), where k and m are natural numbers and $s \in S$.

Kunze discovered that $BR(S,\theta)$ is the Zappa-Szép product of $(N,+)$ and semidirect product, $N \rtimes S$, where multiplication in $N \rtimes S$ is defined by the following rule

$$(k,s)\cdot(l,t) = (k+l,(s\theta^{l})t).$$

Define for $m \in N$ and $(l,s) \in N \rtimes S$

$$m\cdot(l,s) = (g-m, s\theta^{g-l}) \text{ and } m^{(l,s)} = g-l$$

where g is greater of m and l. Then $(N \rtimes S) \otimes N$ is Zappa-Szép product with composition rule

$$[(k,s),m]\circ[(l,t),n] = [(k-m+g, s\theta^{g-m}t\theta^{g-l}), n-l+g],$$

where again g is greater of m and l.

Applications of Zappa-Szép product to semigroups

Rees matrix semigroups

Consider a Rees matrix semigroup $M^0[G, I, \Lambda, P]$. *Kunze* canonically extended $M^0[G, I, \Lambda, P]$ to a Rees matrix semigroup $M^0[G, I', \Lambda', P']$, where $I' = I \cup \{*\}$, $\Lambda' = \Lambda \cup \{*\}$ and

$$p'_{\lambda i} = \begin{cases} p_{\lambda i} & \text{if} \quad \lambda \in \Lambda, i \in I \\ 1 & \text{otherwise} \end{cases}$$

Then $S = \{(i, 1, *) \mid i \in I'\}$ and $T = \{(*, a, \lambda) \mid a \in G, \lambda \in \Lambda'\}$ are subsemigroups of $M^0[G, I', \Lambda', P']$, because $p_{\star j} = 1$.
Now if I' is left zerosemigroup and Λ' is right zero semigroup, then clearly $S \cong I'$ and $T \cong G \times \Lambda'$, because $p_{\lambda *} = 1$.
Furthermore every element $(i, a, \lambda) \in M^0[G, I', \Lambda', P'] \setminus \{0\}$ is uniquely represented as a product in $S \cdot T$.
Thus $M^0[G, I', \Lambda', P']$ is a Zappa-Szép product of S and T, with action of S on T and T on S defined by
$(a, \lambda)^j = (ap_{\lambda j,*})$ and $(a, \lambda).j = *$ respectively.

Congruences in Zappa-Szép products

Proposition 1

Let $S \bowtie T$ be a Zappa-Szép product of semigroups S and T. Let $\sim_1$, $\sim_2$ be congruences on S and T respectively. Define $\sim$ on $S \bowtie T$ by

$$(s,t) \sim (s',t') \Leftrightarrow s \sim_1 s' \wedge t \sim_2 t'.$$

Then $\sim$ is a congruence on $S \bowtie T$ if

$$s \sim_1 s' \wedge t \sim_2 t' \Rightarrow \begin{cases} t \cdot s \sim t' \cdot s' & \text{and} \\ t^s \sim t'^{s'} \end{cases} \longrightarrow \star$$

For Zappa-Szép product of monoids the converse is also true.

If $\star$ is valid, then $(S \bowtie T/\sim) \cong (S/\sim) \bowtie (T/\sim)$ for appropriately defined functions (1) and (2) on $(S/\sim) \bowtie (T/\sim)$.

Congruences in Zappa-Szép products

Proposition 2

Let $S \bowtie T$ be a Zappa-Szép product of monoids and $\sim$ a congruence relation on T such that

$$t \sim t' \Rightarrow t^s \sim t'^{s'} \text{ for every } s \in S.$$

Define an equivalence on S by

$$s \approx s' \Leftrightarrow \forall t \in T : t^s \sim t'^{s'}.$$

If $T/\sim$ is right cancellative, $(s,t) \sim (s',t') \Leftrightarrow s \approx s' \wedge t \sim t'$ is a congruence on $S \bowtie T$.

An interesting situation

Example

As an example for a situation dual to above proposition take $T = A^*$ and S as the set of states for an accepting automaton $\mathcal{A} = (A, S, \delta, q_0, F)$ for a language $L \subseteq A^*$. Equip S with right zero multiplication to fulfill additional assumption of Proposition (2). Put

$$a^s = 1, a.s = \delta(a, s)$$

where δ is state transition function of $\mathcal{A}$, and consider an equivalence $\sim$ on S satisfying

$$s_1 \sim s_2 \text{ implies } \delta(w, s_1) \in F \Leftrightarrow \delta(w, s_2) \in F$$

for every $w \in A^*$.
If $\sim$ is equality, then $A^*/\approx$ is the transition monoid of $\mathcal{A}$.
If $\sim$ is coarsest possible equivalence, then $A^*/\approx$ is the syntactic monoid of L.

Regular Zappa-Szép product

- The (internal) Zappa-Szép product $M = A \bowtie B$ of the regular subsemigroups (submonoids) A and B need not be regular in general. Following is an example related to this situation.

Example

Let $A = \{1, e, f\}$ where $e^2 = e, f^2 = f, ef = f = fe$ and $B = \{1, b\}$ where $b^2 = b$. Suppose $1 \in A$ act trivially on B, so that

$$1^e = 1^f = 1, b^e = b^f = 1$$

and $1 \in B$ act trivially on A, that is

$$b \cdot 1 = 1, b \cdot e = f, b \cdot f = f.$$

Then A and B are regular monoids.

We have following multiplication table:

	$(1,1)$	$(1,b)$	$(e,1)$	(e,b)	$(f,1)$	(f,b)
$(1,1)$	$(1,1)$	$(1,b)$	$(e,1)$	(e,b)	$(f,1)$	(f,b)
$(1,b)$	$(1,b)$	$(1,b)$	$(f,1)$	(f,b)	$(f,1)$	(f,b)
$(e,1)$	$(e,1)$	(e,b)	$(e,1)$	(e,b)	$(f,1)$	(f,b)
(e,b)	(e,b)	(e,b)	$(f,1)$	(f,b)	$(f,1)$	(f,b)
$(f,1)$	$(f,1)$	(f,b)	$(f,1)$	(f,b)	$(f,1)$	(f,b)
(f,b)	(f,b)	(f,b)	$(f,1)$	(f,b)	$(f,1)$	(f,b)

From the table we see that multiplication is associative. Then $M = A \bowtie B$ is Zappa-Szép product of A and B which is not regular, since (e,b) is not a regular element. Moreover

$$E(M) = \{(1,1),(1,b),(e,1),(f,1),(f,b)\}$$

which is not a subsemigroup of M, since $(e,1)(1,b) = (e,b) \notin E(M)$.

Regular Zappa-Szép product

Proposition

If A is regular monoid, B is a group, $1_B \cdot a = a, (1_B)^a = 1_B$, for all $a \in A$, then $M = A \bowtie B$ is regular.

Proposition

Let A be a left zero semigroup and B be a regular semigroup. Suppose for all $b \in B$, there exists some $a \in A$ such that $b^a = b$, and for all $x \in A$, there exists some $b' \in V(b)$ such that $(b')^x = b'$. Then $M = A \bowtie B$ is regular.

Inverse semidirect product

The semidirect product of inverse semigroups/monoids need not be inverse in general. We see it from the following example.

Example Let $S = \{1, a\}$ be the commutative monoid with one non zero-identity idempotent a. Let $T = \{1, e, 0\}$ be the commutative monoid with zero and $e = e^2$.

Then S and T are both inverse monoids, and there is a homomorphism

$$\theta : S \to End(T) \text{ given by}$$
$$1^a = 1, e^a = 0^a = e.$$

Then $P = S \ltimes T$ is regular. However the element $(a, e) \in P$ has two inverses, (a, e) and $(a, 0)$.

Hence P is not inverse monoid.

Inverse semidirect product

A complete characterization of semidirect product of inverse monoids is given by *W.R. Nico* in 1983.

Theorem

A semidirect product $P = S \ltimes T$ of two monoids S and T determined by the homomorphism $\theta : S \rightarrow End(T)$ will be an inverse monoid if and only if,

(i) S and T are inverse monoids.

(ii) For every $e = e^2 \in S, \theta(e) = 1 \in End(T)$, i.e $t^e = t$ for all $t \in T$.

The Zappa-Szép product $P = S \bowtie T$ of inverse semigroups S and T need not be inverse in general as we we can see from following example.

Example

Let $S = \{e, f, ef\}$, where $e^2 = e, f^2 = f, ef = fe$ and $T = \{1, a, b, ab\}$, where $a^2 = 1, b^2 = 1, ab = ba$.

Suppose that action of T on S defined by

$$a \cdot e = f, b \cdot e = f, a \cdot f = e, b \cdot f = e$$

and $1 \in T$ act trivially. The action of S on T defined by

$$a^e = f, b^e = b, a^f = b, b^f = b.$$

Thus S and T are inverse but $P = S \bowtie T$ is not inverse.

Inverse Zappa-Szép product

Suha wazan has given necessary condition for the Zappa-Szép product of inverse semigroups to be inverse in the following Proposition:

Proposition

The Zappa-Szép product $P = S \bowtie T$ of S and T is inverse semigroup if and only if,
(i) S and T are inverse semigroups,
(ii) $E(S)$ and $E(T)$ act trivially,
(iii) If $p = st \in E(P)$, then s and t act trivially on each other.